I. Measurement Error

Step-up : centered Data $\qquad y_i = X_i\beta + \varepsilon_i \qquad \varepsilon_i \sim iid(0, \sigma_\varepsilon^2)$

Observe : $\{y_i^*, X_i^*\}$ $\qquad y_i^* = y_i + \eta_i \qquad \eta_i \sim iid(0, \sigma_\eta^2)$

$\qquad X_i^* = X_i + v_i \qquad v_i \sim iid(0, \sigma_v^2)$

Assumption : A2.2. $\{y_i, X_i\}$ JS & E.

X_i independent of all error terms $\varepsilon_i, v_i, \eta_i$ $\quad$ & $V(X_i) = \sigma_x^2$

y_i 〃 〃 η_i, v_i

All errors are mutually independent. $\varepsilon \perp v \perp \eta \perp \varepsilon$

Reg y_i^* on X_i^*

$$\hat\beta_{OLS} = \frac{\sum y_i^* x_i^*}{\sum X_i^*} = \frac{\sum (y_i - \eta_i)(X_i - v_i)}{\sum (X_i - v_i)^2} = \frac{\sum (X_i\beta + \varepsilon_i - \eta_i)(X_i - v_i)}{\sum (X_i - v_i)^2}$$

$$= \frac{\beta \overbrace{\sum X_i^2}^{\xrightarrow{p}\sigma_x^2} - \beta \overbrace{\sum X_i v_i}^{\xrightarrow{p}0} + \overbrace{\sum \varepsilon_i X_i}^{\xrightarrow{p}0} - \overbrace{\sum \varepsilon_i v_i}^{\xrightarrow{p}0} - \overbrace{\sum \eta_i X_i}^{\xrightarrow{p}0} + \overbrace{\sum \eta_i v_i}^{\xrightarrow{p}0}}{\underbrace{\sum X_i^2}_{\xrightarrow{p}\sigma_x^2} + \underbrace{\sum v_i^2}_{\xrightarrow{p}\sigma_v^2} + 2\underbrace{\sum X_i v_i}_{\xrightarrow{p}0}}$$

$$\text{plim}\ \hat\beta_{OLS} = \beta \frac{\sigma_x^2}{\sigma_x^2 + \sigma_v^2}$$

Case ①: $\sigma_\eta^2 > 0$ & $\sigma_v^2 = 0$ $\qquad$ i.e. only y_i has measurement error.

$$\text{plim}\ \hat\beta_{OLS} = \beta \frac{\sigma_x^2}{\sigma_x^2 + 0} = \beta \qquad \text{consistent.}$$

Intuition : measurement error in y_i can be "included" into the regression error. $\quad$ i.e. $\varepsilon_i^* = \varepsilon_i + \eta_i$

Case ②: $\sigma_v^2 > 0$

$$\text{plim}\ \hat\beta_{OLS} = \beta \frac{\sigma_x^2}{\sigma_x^2 + \sigma_v^2} < \beta \qquad \text{"attenuation bias"}$$

Claim 1: $\overset{n\times n}{A}, \overset{n\times n}{B}$ nonsingular

Then AB & BA have the same e.values

pf:

A & B full rank $\Rightarrow$ AB full rank

$AB = C\Lambda C^{-1}$ (Note: symmetric matrix has $C\Lambda C'$)

$BA = BABB^{-1} = BC\Lambda C^{-1}B^{-1} = \underbrace{(BC)}_{E}\Lambda(BC)^{-1}$

BA has e.value matrix Λ. ▨

Claim 2: $\underset{n\times n}{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & O \\ O & & \ddots \\ & & & \lambda_n \end{bmatrix}$ $[\Lambda - I_n$ is p.s.d$]$

$\Leftrightarrow [\lambda_i \geq 1 \quad i=1,\cdots,n]$

pf: Note: Consider $b = (b_1, b_2, \cdots, b_n)'$

$$b'(\Lambda - I_n)b = \sum_{i=1}^n b_i^2(\lambda_i - 1)$$

"$\Rightarrow$" Suppose $\lambda_i < 1$ for some i. Let $b = (0,\cdots,0,1,0,\cdots,0)$ ↳ i^{th} position

$b'(\Lambda - I_n)b = \lambda_i - 1 < 0$ ↯

"$\Leftarrow$" Take any $b \in \mathbb{R}^n$. $b \neq \vec{0}$

$$b'(\Lambda - I_n)b = \sum_{i=1}^n b_i^2 \underbrace{(\lambda_i - 1)}_{\geq 0 \ \forall i} \geq 0$$ ▨

Claim 3: If $\overset{n\times n}{A} - \overset{n\times n}{B}$ is p.s.d. & $\overset{n\times n}{C}$ is nonsingular,

then $C'(A-B)C$ is p.s.d.

pf:

Take $x \in \mathbb{R}^n$ $x \neq \vec{0}$.

Since C is nonsingular, $Cx \in \mathbb{R}^n$ & $Cx \neq \vec{0}$

Since $A-B$ is p.s.d.

$$(Cx)'(A-B)Cx \geq 0$$

$$x'C'(A-B)Cx \geq 0$$

Since x is arbitrarily taken, $C'(A-B)C$ is p.s.d. ▨

<u>Claim 4</u> : If $\overset{n\times n}{A}$ is symmetric, then $A = C\Lambda C'$,
where $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is e.value matrix & $C^{-1} = C'$.

Proved in Recitation 1.

<u>Claim</u> : $\overset{k\times k}{A}$ & $\overset{k\times k}{B}$ are symmetric, p.d., nonsingular matrices
$A - B$ p.s.d $\Rightarrow$ $B^{-1} - A^{-1}$ p.s.d

<u>proof</u> :

A & B p.d $\Rightarrow$ $A = C\Lambda_1 C' = P_A P_A'$ w/ $P_A = C\Lambda_1^{\frac{1}{2}}$
$B = D\Lambda_2 D' = P_B P_B'$ w/ $P_B = D\Lambda_2^{\frac{1}{2}}$

where P_A & P_B are nonsingular. $\Rightarrow$ P_A^{-1} & P_B^{-1} nonsingular

Note: $P_B^{-1}(A-B)P_B^{-1\prime} = P_B^{-1}AP_B^{-1\prime} - P_B^{-1}P_BP_B'P_B'^{-1} = P_B^{-1}AP_B^{-1\prime} - I_k$

By Claim 3, $P_B^{-1}AP_B^{-1\prime} - I_k$ is p.s.d.

Note: $P_B^{-1}AP_B^{-1\prime}$ is symmetric:

$$[P_B^{-1}AP_B^{-1\prime}]_{ij} = \sum_{k=1}^{n}\sum_{\ell=1}^{n}[P_B^{-1}]_{ik}[A_{k\ell}][P_B^{-1\prime}]_{\ell j}$$

$$= \sum_{\ell=1}^{n}\sum_{k=1}^{n}[P_B^{-1}]_{j\ell}[A_{\ell k}][P_B^{-1\prime}]_{ki} = [P_B^{-1}AP_B^{-1\prime}]_{ji}$$

So, by spectral decomposition, $P_B^{-1}AP_B^{-1\prime} = C^*\Lambda C^{*\prime}$ w/ $C^{*\prime} = C^{*-1}$

$\Rightarrow$ $C^*\Lambda C^{*\prime} - I_k$ is p.s.d.

$$C^{*\prime}(C^*\Lambda C^{*\prime} - I_k)C^* = C^{*\prime}C^*\Lambda C^{*\prime}C^* - C^{*\prime}I_kC^* = \Lambda - I_k$$

By Claim 3, $\Lambda - I_k$ is p.s.d.

By Claim 2: $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_k \end{bmatrix}$ $\lambda_i \geq 1$ $i = 1, \cdots, k$

Note: $P_B^{-1} A P_B^{-1'} = P_B^{-1} P_A P_A' P_B^{-1'} = (P_B^{-1} P_A)(P_B^{-1} P_A)'$

By Claim 1, $(P_B^{-1} P_A)(P_B^{-1} P_A)'$ & $(P_B^{-1} P_A)'(P_B^{-1} P_A)$ have the same e. values.

Note: $(P_B^{-1} P_A)'(P_B^{-1} P_A) = P_A' P_B^{-1'} P_B^{-1} P_A = P_A'(P_B P_B')^{-1} P_A = P_A' B^{-1} P_A$

Also, $P_A' B^{-1} P_A$ is symmetric.

By spectral decomposition: $P_A' B^{-1} P_A = F \Lambda F'$ where $F' = F^{-1}$

↳ the same e.values

Recall: $\lambda_i \geq 1$ $i = 1, 2, \cdots, k$ $\Lambda - I_k$ is p.s.d.

$$F(\Lambda - I_k)F' = F\Lambda F' - FF' = P_A' B^{-1} P_A - I_k$$

By Claim 3: $P_A' B^{-1} P_A - I_k$ is p.s.d.

$$P_A^{-1'}(P_A' B^{-1} P_A - I_k) P_A^{-1}$$

$$= P_A'^{-1} P_A B^{-1} P_A P_A^{-1} - P_A'^{-1} P_A^{-1} = B^{-1} - (P_A P_A')^{-1} = B^{-1} - A^{-1}$$

By Claim 3: $B^{-1} - A^{-1}$ is p.s.d. $\blacksquare$