I. Thm 5.2.1. (Fuller)

II. Delta method.

I.

<u>Claim</u>: X, Y : RVs w/ cdf $F_X(\cdot)$ & $F_Y(\cdot)$.

Fix $\varepsilon > 0$ & $\delta > 0$.

$$\Pr\{|X - Y| > \varepsilon\} < \delta \Rightarrow F_X(x-\varepsilon) - \delta < F_Y(x) < F_X(x+\varepsilon) + \delta \quad \forall$$

<u>pf</u>:

Take $\varepsilon > 0$, $\delta > 0$. Suppose $\Pr\{|X-Y| > \varepsilon\} < \delta$

$$\begin{aligned}
F_X(x-\varepsilon) - F_Y(x) &= \Pr\{X \le x - \varepsilon\} - \Pr\{Y \le x\} \\
&\le \Pr\{\underbrace{X \le x - \varepsilon}_{A}\} - \Pr\{X \le x - \varepsilon \wedge \underbrace{Y \le x}_{B}\} \\
&= \Pr\{\underbrace{X \le x - \varepsilon}_{A} \wedge \underbrace{Y > x}_{C}\} \\
&\le \Pr\{\underbrace{Y - X > \varepsilon}_{D}\} \\
&\le \Pr\{|Y - X| > \varepsilon\} \\
&< \delta
\end{aligned}$$

$\Pr\{A\} = \Pr\{A \wedge B\} + \Pr\{A \wedge \neg B\}$

$A \wedge C \Rightarrow D$

$\Pr\{A \wedge C\} \le \Pr\{D\}$

since $\varepsilon > 0$

by assumption

$$\Rightarrow F_X(x-\varepsilon) - \delta < F_Y(x)$$

$$\begin{aligned}
F_Y(x) - F_X(x+\varepsilon) &= \Pr\{Y \le x\} - \Pr\{X \le x + \varepsilon\} \\
&\le \Pr\{Y \le x\} - \Pr\{Y \le x \wedge X \le x + \varepsilon\} \\
&= \Pr\{Y \le x \wedge X > x + \varepsilon\} \\
&\le \Pr\{X - Y > \varepsilon\} \\
&\le \Pr\{|X - Y| > \varepsilon\} \\
&< \delta
\end{aligned}$$

$$\Rightarrow F_Y(x) < F_X(x+\varepsilon) + \delta$$

Therefore, $F_X(x-\varepsilon) - \delta < F_Y(x) < F_X(x+\varepsilon) + \delta$ ▨

Thm 5.2.1.

Let $\{X_n\}$ & $\{Y_n\}$ be sequences of random variables s.t.

$$\operatorname*{plim}_{n\to\infty} (X_n - Y_n) = 0$$

If there exists a random variable X s.t. $X_n \xrightarrow{D} X$

then $Y_n \xrightarrow{D} X$

pf :

$F_X(\cdot)$: cdf of RV X.

Let x_0 be any continuity point for $F_X(\cdot)$.

Take any $\delta > 0$, $\exists\, \eta > 0$ s.t.

$$|x - x_0| \le \eta \wedge F_X(\cdot) \text{ is cts. at } x_0 - \eta \ \& \ x_0 + \eta \quad \Rightarrow |F_X(x) - F_X(x_0)| < \tfrac{1}{4}\delta$$

Since $\operatorname*{plim}_{n\to\infty} (X_n - Y_n) = 0$,

$$\exists\, N(\eta, \tfrac{\delta}{2}) \quad \text{s.t.} \quad Pr\{|X_n - Y_n| > \eta\} < \frac{\delta}{2} \quad \forall n > N(\eta, \tfrac{\delta}{2})$$

$$\Rightarrow \text{By Claim,} \quad F_{X_n}(x - \eta) - \tfrac{1}{2}\delta < F_{Y_n}(x) < F_{X_n}(x + \eta) + \tfrac{1}{2}\delta \quad \forall x \quad (*)$$

Since $X_n \xrightarrow{D} X$. i.e. $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all x for which $F_X(\cdot)$ is continuous

$$\exists\, M_1 \ \& \ M_2 \quad \text{s.t.} \quad |F_{X_n}(x_0 - \eta) - F_X(x_0 - \eta)| < \tfrac{1}{4}\delta \quad \forall n > M_1$$

$$\& \quad |F_{X_n}(x_0 + \eta) - F_X(x_0 + \eta)| < \tfrac{1}{4}\delta \quad \forall n > M_2$$

Let $N = \max\{M_1, M_2, N(\eta, \frac{\delta}{2})\}$

$\forall n > N$ we have

$$F_X(x_0) - \delta \underset{\substack{\downarrow \\ \text{continuity} \\ \text{of } F_X(\cdot)}}{<} F_X(x_0 - \eta) - \tfrac{3}{4}\delta \underset{\substack{\downarrow \\ X_n \xrightarrow{D} X}}{<} F_{X_n}(x_0 - \eta) - \tfrac{1}{2}\delta \underset{\substack{\downarrow \\ (*)}}{<} F_{Y_n}(x_0)$$

$$F_X(x_0) + \delta > F_X(x_0 + \eta) + \tfrac{3}{4}\delta > F_{X_n}(x_0 + \eta) + \tfrac{1}{2}\delta > F_{Y_n}(x_0)$$

$$\Rightarrow |F_{Y_n}(x_0) - F_X(x_0)| < \delta \quad \forall n > N \qquad \text{i.e.} \ \lim_{n\to\infty} F_{Y_n}(x_0) = F_X(x_0) \qquad Y_n \xrightarrow{D} X.$$

$\square$

II. Delta Method.

If $\underset{k\times 1}{\hat{\beta}_n} \xrightarrow{P} \underset{k\times 1}{\beta}$ (random vectors) & $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{D} \underset{k\times 1}{W} \sim N(0, \underset{k\times k}{\Sigma})$

$g(\cdot): \mathbb{R}^k \to \mathbb{R}^p$ continuously differentiable

($\hookrightarrow$ first derivative is continuous)

$$\underset{p\times k}{\Gamma(\theta)} \equiv \frac{\partial g(\theta)}{\partial \theta'} = \begin{bmatrix} \frac{\partial g_1(\theta)}{\partial \theta_1} & \frac{\partial g_1(\theta)}{\partial \theta_2} & \cdots & \frac{\partial g_1(\theta)}{\partial \theta_k} \\ \frac{\partial g_2(\theta)}{\partial \theta_1} & \frac{\partial g_2(\theta)}{\partial \theta_2} & \cdots & \frac{\partial g_2(\theta)}{\partial \theta_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p(\theta)}{\partial \theta_1} & \frac{\partial g_p(\theta)}{\partial \theta_2} & \cdots & \frac{\partial g_p(\theta)}{\partial \theta_k} \end{bmatrix}$$

Jacobian matrix

Then, for a specific value of β,

$$\sqrt{n}(g(\hat{\beta}) - g(\beta)) \xrightarrow{D} \underset{p\times k}{\Gamma(\beta)}\underset{k\times 1}{W} \sim N(0, \underset{p\times k}{\Gamma(\beta)}\underset{k\times k}{\Sigma}\underset{k\times p}{\Gamma(\beta)'})$$

pf:

Fix β. Take $m \in \{1, 2, \cdots, p\}$, $\ell \in \{1, 2, \cdots, k\}$, $n \in \mathbb{N}$

By Mean Value Thm, $\exists\, c \in (0,1)$ s.t.

$$\text{s.t. } g_m(\hat{\beta}_n) - g_m(\beta) = \nabla g_m(\underbrace{(1-c)\beta + c\hat{\beta}_n}_{\tilde{\beta}_{n,m} = (\tilde{\beta}_{n,m1}, \cdots, \tilde{\beta}_{n,m\ell})}) \cdot (\hat{\beta}_n - \beta)$$

$$= \sum_{\ell=1}^{k} \left.\frac{\partial g_m(\theta)}{\partial \theta_\ell}\right|_{\theta = \tilde{\beta}_{n,m}} (\hat{\beta}_{n\ell} - \beta_\ell)$$

$$= \sum_{\ell=1}^{k} [\Gamma(\tilde{\beta}_{n,m})]_{m\ell} (\hat{\beta}_{n\ell} - \beta_\ell)$$

$$= \underbrace{[\Gamma(\tilde{\beta}_{n,m})]_m}_{\substack{\text{the } m^{th} \text{ row} \\ 1\times k}} \underset{k\times 1}{(\hat{\beta}_n - \beta)}$$

$$g(\hat{\beta}_n) - g(\beta) = \begin{bmatrix} g_1(\hat{\beta}_n) - g_1(\beta) \\ \vdots \\ g_p(\hat{\beta}_n) - g_p(\beta) \end{bmatrix} = \underbrace{\begin{bmatrix} [\Gamma(\tilde{\beta}_{n,1})]_1 \\ [\Gamma(\tilde{\beta}_{n,2})]_2 \\ \vdots \\ [\Gamma(\tilde{\beta}_{n,p})]_p \end{bmatrix}}_{p \times k} \underbrace{(\hat{\beta}_n - \beta)}_{k \times 1}$$

$$\sqrt{n}\left(g(\hat{\beta}_n) - g(\beta)\right) = \begin{bmatrix} [\Gamma(\tilde{\beta}_{n,1})]_1 \\ \vdots \\ [\Gamma(\tilde{\beta}_{n,p})]_p \end{bmatrix} \left(\sqrt{n}(\hat{\beta}_n - \beta)\right) \qquad (*)$$

$$\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{D} W \sim N(0, \Sigma)$$

Note: $\hat{\beta}_n \xrightarrow{P} \beta \Rightarrow \hat{\beta}_{n\ell} \xrightarrow{P} \beta_\ell \quad \ell = 1, 2, \cdots, k$

$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \Pr\{|\hat{\beta}_{n\ell} - \beta_\ell| > \varepsilon\} = 0$

Since $\forall m \in \{1, \cdots, p\}$, $\tilde{\beta}_{n,m\ell}$ is between β_ℓ & $\hat{\beta}_{n\ell}$,

$\lim_{n\to\infty} \Pr\{|\tilde{\beta}_{n,m\ell} - \beta_\ell| > \varepsilon\} = 0 \quad \forall \varepsilon > 0 \quad \Rightarrow \tilde{\beta}_{n,m\ell} \xrightarrow{P} \beta_\ell \quad \ell = 1, 2, \cdots, k; \; m = 1, 2, \cdots, p$

$\Rightarrow \forall m = 1, 2, \cdots, p, \quad \tilde{\beta}_{n,m} \xrightarrow{P} \beta$

fixed, so β is a vector of constants

Since $[\Gamma(\tilde{\beta}_{n,m})]_m$ is cts in $\tilde{\beta}_{n,m}$

$\Rightarrow [\Gamma(\tilde{\beta}_{n,m})]_m \xrightarrow{P} [\Gamma(\beta)]_m \quad \forall m$

$$\begin{bmatrix} [\Gamma(\tilde{\beta}_{n,1})]_1 \\ \vdots \\ [\Gamma(\tilde{\beta}_{n,p})]_p \end{bmatrix} \xrightarrow{P} \begin{bmatrix} [\Gamma(\beta)]_1 \\ \vdots \\ [\Gamma(\beta)]_p \end{bmatrix} = \Gamma(\beta)$$

a matrix of constant

Recall from Lecture Notes :

$$\underset{k\times 1}{X_n} \xrightarrow{D} \underset{k\times 1}{X} \quad \& \quad \underset{p\times k}{Y_n} \xrightarrow{P} \underset{p\times k}{\overset{\text{a matrix of constants}}{Y}} \quad \Rightarrow \quad Y_n X_n \xrightarrow{D} YX$$

Consider (*)

$$\sqrt{n}\left(g(\hat{\beta}_n) - g(\beta)\right) = \underbrace{\begin{bmatrix} [\Gamma(\tilde{\beta}_{n,1})]_1 \\ \vdots \\ [\Gamma(\tilde{\beta}_{n,p})]_p \end{bmatrix}}_{\xrightarrow{P}\ \Gamma(\beta)} \underbrace{\left(\sqrt{n}(\hat{\beta}_n - \beta)\right)}_{\xrightarrow{D}\ W \sim N(0,\Sigma)}$$

$$\Rightarrow \sqrt{n}\left(g(\hat{\beta}_n) - g(\beta)\right) \xrightarrow{D} \Gamma(\beta) W \sim N(0, \Gamma(\beta)\Sigma\Gamma(\beta)')$$

$\blacksquare$